Chebyshev lattices as unifying framework for point sets used in interpolation and integration

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Overview

1. Generalisation to s-D
2. Chebyshev lattice rules
3. Classification existing rules
4. Search
5. Results
6. Conclusion

1. Generalisation to s-D

1 dimensional: $f(x), x \in [-1, 1]$

- Chebyshev series $f(x) = \sum_{h=0}^{\infty} \alpha_h \, \hat{T}_h(x)$

- Interpolating approx. $f(x) \approx \sum_{h=0}^{n} a_h \, \hat{T}_h(x)$

- Quadrature rule (Clenshaw-Curtis) $\int_{-1}^{1} f(x) \, \mathrm{d}x \approx \sum_{h=0}^{n} a_h \, I_{[\hat{T}_h]}$

1. Generalisation to s-D

s-dimensional: $f(\vec{x}), \vec{x} \in [-1,1]^s$

- Chebyshev series $f(\vec{x}) = \sum_{\vec{h}} \alpha_{\vec{h}} \hat{T}_{\vec{h}}(\vec{x})$

- Hyperinterpolation $f(\vec{x}) \approx \sum_{\vec{h},|\vec{h}|\leq n} a_{\vec{h}} \hat{T}_{\vec{h}}(\vec{x})$

- Cubature rule $\int_{[-1,1]^s} f(\vec{x})\,\mathrm{d}\vec{x} \approx \sum_{\vec{h},|\vec{h}|\leq n} a_{\vec{h}} I_{[\hat{T}_{\vec{h}}]}$

1. Generalisation to s-D

HYPERINTERPOLATION

$$f(\vec{x}) \approx \sum_{\vec{h},|\vec{h}|\leq n} a_{\vec{h}}\, \hat{T}_{\vec{h}}(\vec{x})$$

Projection into polynomial function space

$$a_{\vec{h}} \approx \int_{[-1,1]^s} f(\vec{x})\, \hat{T}_{\vec{h}}(\vec{x})\, \omega(\vec{x})\, \mathrm{d}\vec{x}$$

using cubature rule for weighted setting

$$a_{\vec{h}} = \sum_{l} w_l\, f(\vec{x}_l)\, \hat{T}_{\vec{h}}(\vec{x}_l)$$

1. Generalisation to s-D

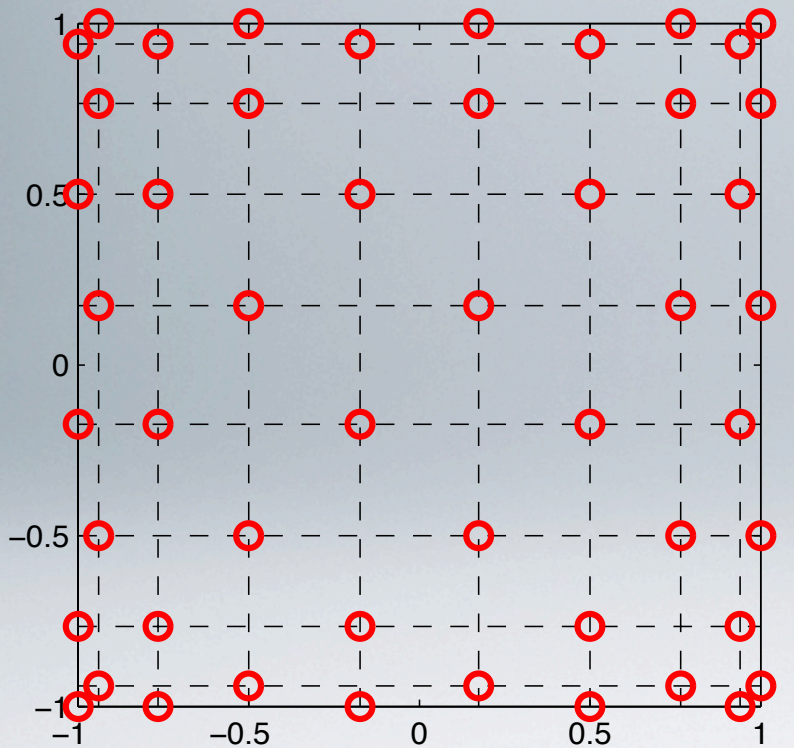
WEIGHTED CUBATURE

$$Q[g] = \sum_{l} w_l \, g(\vec{x}_l) \approx \int_{[-1,1]^s} g(\vec{x}) \, \omega(\vec{x}) \, \mathrm{d}\vec{x}$$

- Several *good* point sets exist
- Known lower bounds on # points
- No clear constructive approach
- Extrema of Chebyshev polynomials

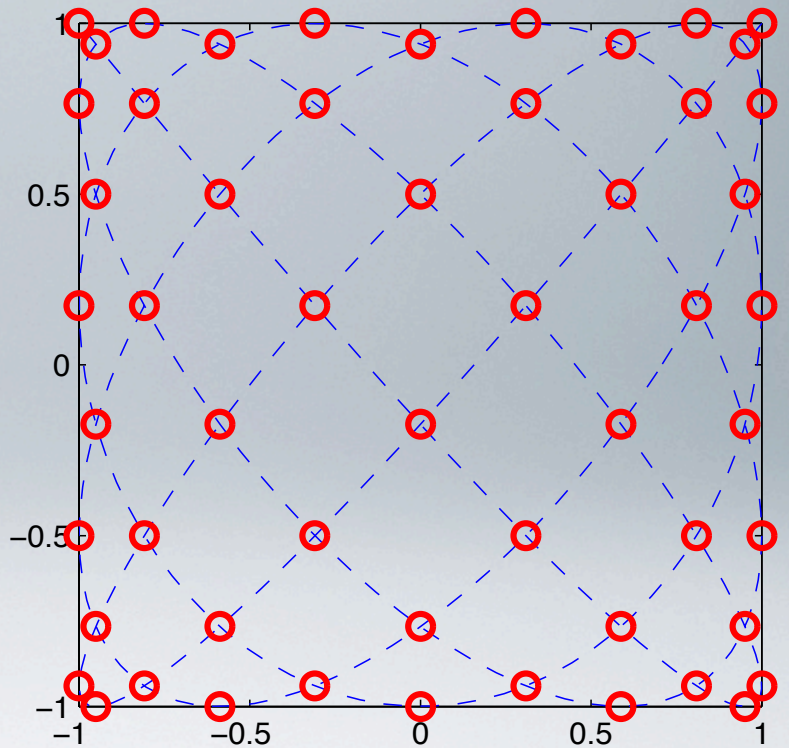
1. Generalisation to s-d

POINT SET EXAMPLES



Morrow-Patterson

(°1978, deg. 17, #50)



Padua

(°2005, deg. 17, #55)

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2. Chebyshev lattice rules

OBJECTIVES

1. Increase number of component values
2. Easy extendible to higher dimensions
3. Compact and uniform representation
4. Enable the use of the FFT
5. Provide a constructive approach

2. Chebyshev lattice rules

CHEBYSHEV LATTICE

$$\vec{x}_{\vec{\ell}} = \cos\left(\pi\left(\frac{\ell_1\,\vec{z}_1}{d_1} + \cdots + \frac{\ell_k\,\vec{z}_k}{d_k} + \frac{\vec{z}_\Delta}{d_\Delta}\right)\right)$$

$$\vec{z}_1, \ldots, \vec{z}_k, \vec{z}_\Delta \in \mathbb{Z}^s \qquad d_1, \ldots, d_k, d_\Delta, \ell_1, \ldots, \ell_k \in \mathbb{Z}$$

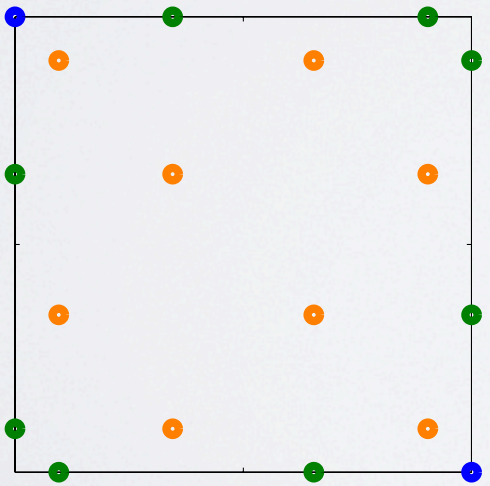
- rank-k integration lattice
- offset vector
- mimic Chebyshev extrema

2. Chebyshev lattice rules

CHEBYSHEV LATTICE RULE

$$\vec{x}_{\vec{\ell}} = \cos\left(\pi\left(\frac{\ell_1\,\vec{z}_1}{d_1} + \cdots + \frac{\ell_k\,\vec{z}_k}{d_k} + \frac{\vec{z}_\Delta}{d_\Delta}\right)\right)$$

$$w_{\vec{\ell}} = \text{cte}\left(\frac{1}{2}\right)^{\#_r(|x_{\vec{\ell},r}|=1)}$$

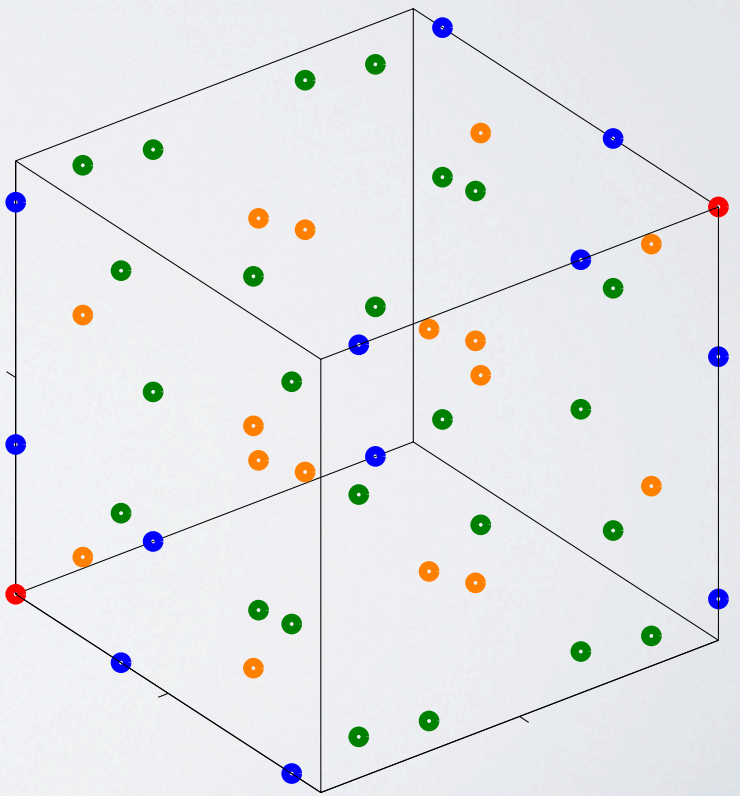


cte

cte/2

cte/4

cte/8



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3. Classification existing rules

Clenshaw-Curtis (1960)

Morrow-Patterson (1978)

n=11, fully symmetric (1988)

Minimal D4-invariant (1989)

Blending, Godzina (1994)

Padua (2005)

Xu (2009)

Padua-like (-)

3. Classification existing rules

	k=s (full rank)	k=s-1
s=1	Clenshaw-Curtis (1960)	—
s=2	Morrow-Patterson (1978) n=11, fully symmetric (1988) Minimal D4-invariant (1989)	Padua (2005)
s=3	Blending, Godzina (1994) *Xu (2009)*	*Padua-like (-)*
s	Blending, Godzina (1994)	*Padua-like (-)*

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4. Search

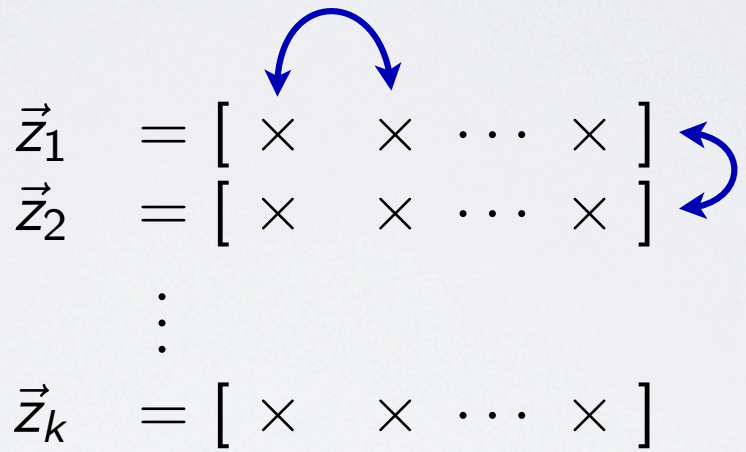
SETUP

- In 2, 3, 4 and 5-D, not limited to rank-1
- Most basic search algorithm:
  - ➡ for each d
    - ➡ for each $\{\vec{z}_1, \ldots, \vec{z}_k\}$
      - create the point set
      - verify its degree
      - store rule if better
- Rank and dimension as parameters

4. Search

SEARCH SPACE REDUCTION

- Size search space: $(d+1)^{ks}$
- Optimisation: exploit symmetries



- How to generate these vectors?

4. Search

SEARCH SPACE GENERATION

- Parametric representation
  - i.e. s=3, rank-1:

$$[\mathtt{A},\mathtt{A},\mathtt{A}]$$
$$[\mathtt{A},\mathtt{A},\mathtt{B}]$$
$$[\mathtt{A},\mathtt{B},\mathtt{B}]$$
$$[\mathtt{A},\mathtt{B},\mathtt{C}]$$

for $\mathtt{A},\mathtt{B},\mathtt{C} = 0,\ldots,d$
$\mathtt{A} < \mathtt{B} < \mathtt{C}$

  - i.e. s=2, rank-2:

$$[\mathtt{A},\mathtt{A}]\ [\mathtt{A},\mathtt{B}]$$
$$[\mathtt{A},\mathtt{B}]\ [\mathtt{A},\mathtt{C}]$$
$$[\mathtt{A},\mathtt{B}]\ [\mathtt{B},\mathtt{A}]$$
$$[\mathtt{A},\mathtt{B}]\ [\mathtt{C},\mathtt{A}]$$
$$\ldots$$

for $\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{D} = 0,\ldots,d$
$\mathtt{A} < \mathtt{B} < \mathtt{C} < \mathtt{D}$

- Parameters through (almost) loopless algorithm

4. Search

POINT SET CREATION

- Problem: duplicates
- Red-black tree
  comparison is bottleneck ...
    - avoid floating point
      - reduce integer parts to $[0, d]$: $\vec{y}_\ell = \text{minmod}\,(\ell\vec{z})$
    - exploit large integer length
      - composite scalar: $y_\ell = \sum_{r=1}^{s} (d+1)^{r-1}\,\text{minmod}\,(\ell z_r)$

$$\vec{x}_\ell = \cos\left(\frac{\pi}{d}\,\ell\vec{z}\right)$$

integer

4. Search

DEGREE VERIFICATION

- Test inner cubature rule for all $\hat{T}_{\vec{h}} \quad (|\vec{h}| \leq n)$ evaluation is bottleneck
  - cache intermediate results
    - trade off: storage vs. computations
  - change order of coefficients $\vec{h}$
    - maximise intermediate cache “hits”
    - (almost) loopless algorithm

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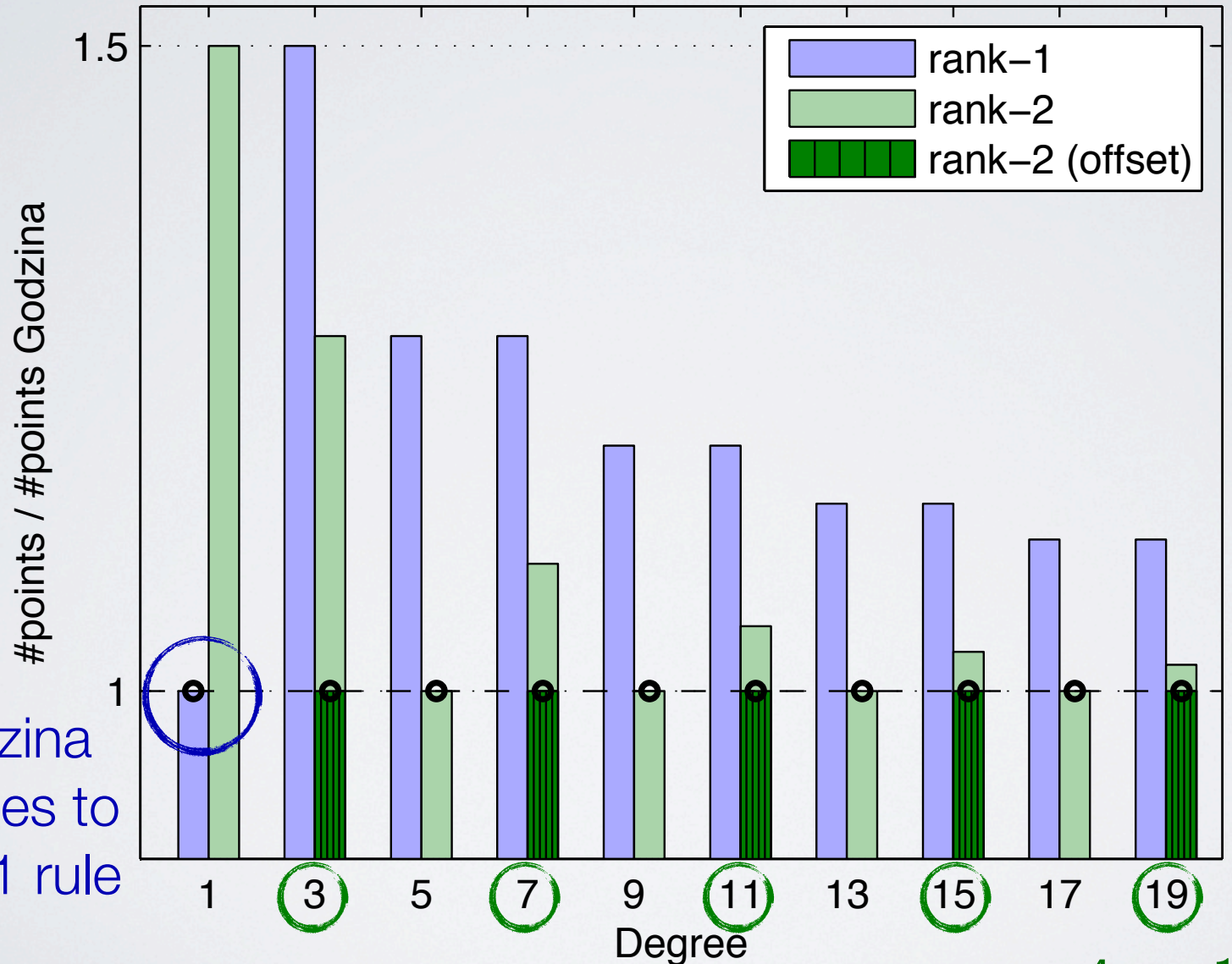
5. Results

GENERAL REMARKS

- Distinct component values still small
  - grid-like structure decreases #points
- No point sets better than already known before
  - Godzina best
- Observations:
  - high rank consistently better than low rank
  - offset vectors often reduce # points

5. Results

2-DIMENSIONS

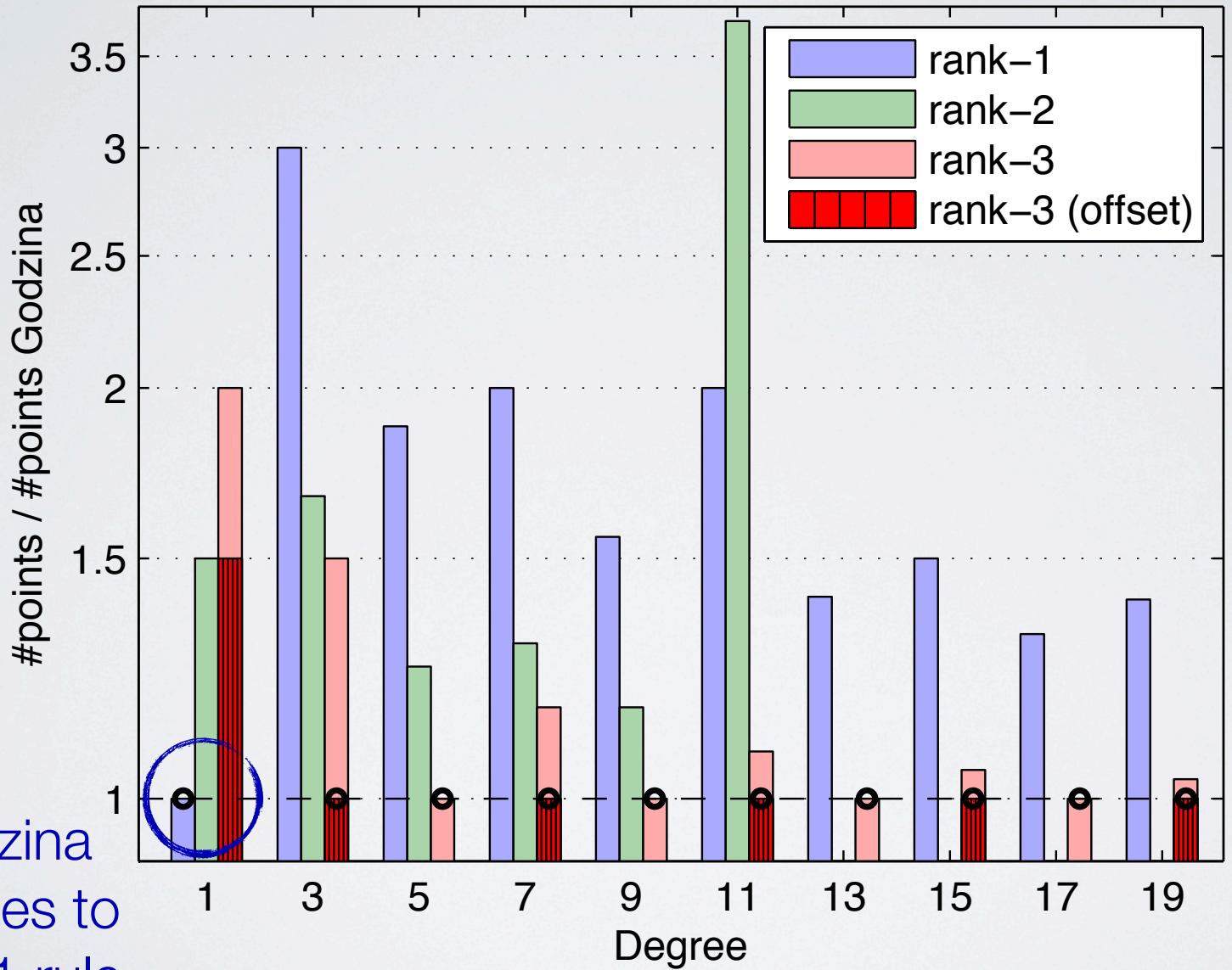


Godzina reduces to rank-1 rule

$n = 4\nu - 1$ with offset

5. Results

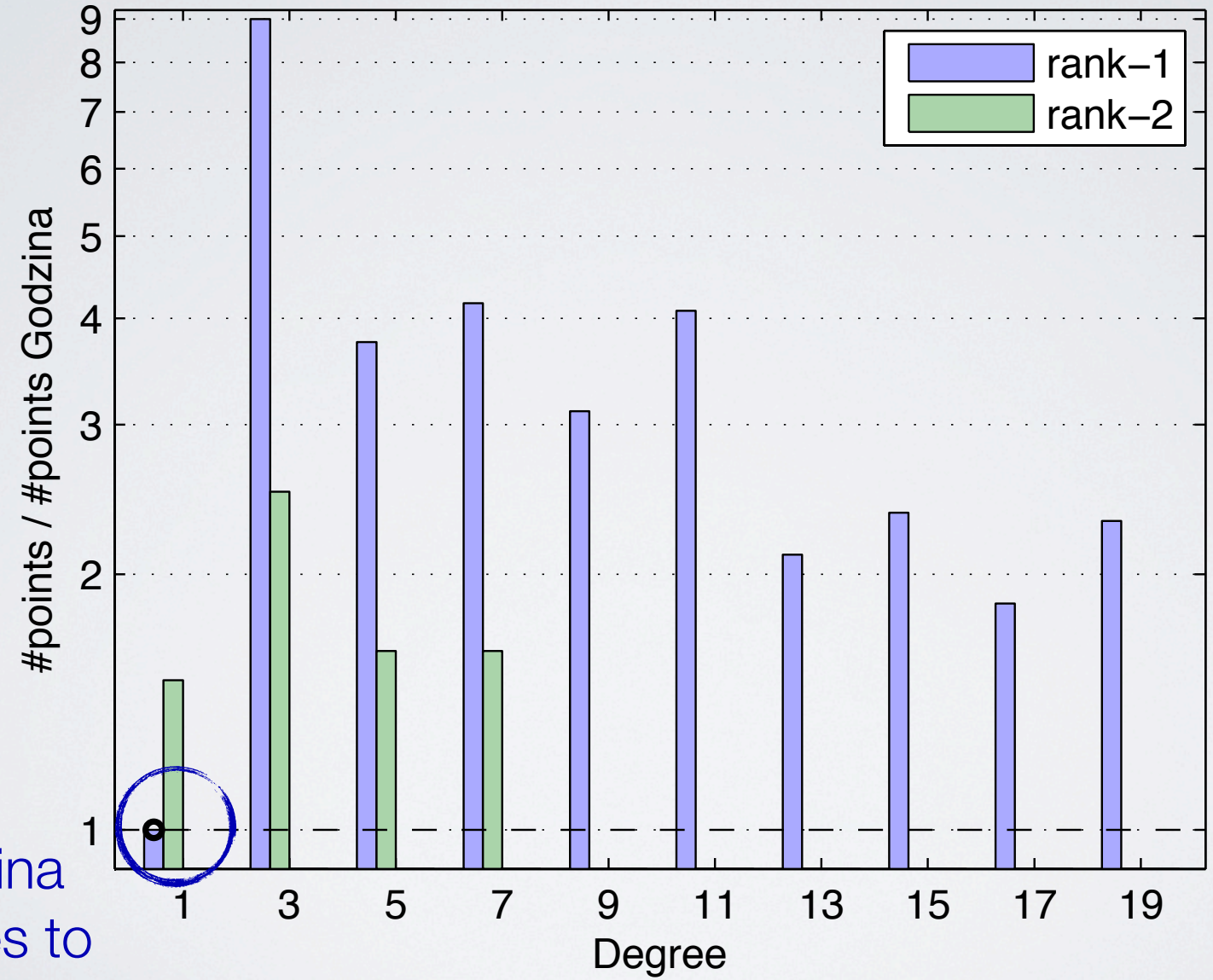
3-DIMENSIONS



Godzina reduces to rank-1 rule

5. Results

4-DIMENSIONS (INCOMPLETE)



Godzina reduces to rank-1 rule

6. Conclusion

- Multivariate extension of Clenshaw-Curtis
- Chebyshev lattices as general framework
- Uniform description for known point sets
- High rank computational search
  - Godzina blending formulae still best
  - Rank-1 rules interesting special case

Future work

- Cubature rules suited for oscillatory functions